

**Dynamical Organization of Cooperation in Complex Topologies**J. Gómez-Gardeñes,<sup>1,2</sup> M. Campillo,<sup>2</sup> L. M. Floría,<sup>1,2</sup> and Y. Moreno<sup>1,\*</sup><sup>1</sup>*Institute for Biocomputation and Physics of Complex Systems (BIFI), University of Zaragoza, Zaragoza 50009, Spain*<sup>2</sup>*Departamento de Física de la Materia Condensada, University of Zaragoza, Zaragoza E-50009, Spain*

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In this Letter, we study how cooperation is organized in complex topologies by analyzing the evolutionary (replicator) dynamics of the Prisoner's Dilemma, a two-players game with two available strategies, defection and cooperation, whose payoff matrix favors defection. We show that, asymptotically, the population is partitioned into three subsets: individuals that always cooperate (*pure cooperators*), always defect (*pure defectors*) and those that intermittently change their strategy. In fact the size of the latter set is the biggest for a wide range of the "stimulus to defect" parameter. While in homogeneous random graphs pure cooperators are grouped into several clusters, in heterogeneous scale-free (SF) networks they always form a single cluster containing the most connected individuals (hubs). Our results give further insights into why cooperation in SF networks is favored.

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To understand the observed survival of cooperation among unrelated individuals in social communities when selfish actions provide a higher benefit, a lot of attention is being paid to the analysis of evolutionary dynamics of simple two-players games like the Prisoner's Dilemma (PD). In this game individuals adopt one of the two available strategies, cooperation or defection; both receive  $R$  under mutual cooperation and  $P$  under mutual defection, while a cooperator receives  $S$  when confronted to a defector, which in turn receives  $T$ , where  $T > R > P > S$ . Under these conditions it is better to defect, regardless of the opponent strategy, and assuming that strategies are allowed to spread within the population according to their payoffs (replicator dynamics [1, 2]), the proportion of cooperators asymptotically vanishes in a well-mixed population (*i.e.* when each agent interacts with all other agents).

If the well-mixed population hypothesis is abandoned, so that individuals only interact with their neighbors in a social network, several studies [3, 4, 5, 6, 7, 8, 9] have reported the asymptotic survival of cooperation on different types of networks. Notably, cooperation even dominates over defection in heterogeneous, SF networks where the distribution density of local connectivities follows a power law. In view of the accumulated evidence [10, 11] that many social (as well as technological, biological and other) networks are highly heterogeneous, these results are highly relevant for the understanding of the evolution of cooperation.

In this Letter, we are interested in exploring the roots of the diverse behavior observed on top of different complex topologies and in providing an explanation in terms of microscopic arguments. We have analyzed in detail the microscopic structural aspects underlying the differences in the evolution of cooperation in a one-parameter family of networks interpolating between Barabási-Albert (BA) [12] and Erdős-Rényi (ER) graphs. As usual in recent studies [3, 4], we choose the Prisoner's Dilemma payoffs as  $R = 1$ ,  $P = S = 0$ , and  $T = b > 1$ , and implement the finite population analogue of replicator dynamics [4]. At each time step  $t$ , which represents

one generation of the discrete evolutionary time, each node  $i$  in the network plays with all its neighbors and accumulates the obtained payoffs,  $P_i$ . Then, the individuals,  $i$ , update synchronously their strategies by picking up at random a neighbor,  $j$ , and comparing their respective payoffs  $P_i$  and  $P_j$ . If  $P_i > P_j$ , nothing happens and  $i$  keeps the same strategy for the next generation. On the contrary, if  $P_j > P_i$ , with probability  $\Pi_{i \rightarrow j} = (P_j - P_i) / \max\{k_i, k_j\}b$ ,  $i$  adopts the strategy of  $j$  for the next round robin with its neighbors [4].

We have performed simulations for a population of  $N$  individuals that interact following the couplings dictated by the underlying graph. To explore the structure and dynamics of cooperative behavior in different topologies, we have made use of the model developed in [13], which allows to smoothly pass from a BA network to a random graph of the sort of ER networks by tuning a single parameter  $\alpha \in (0, 1)$ . We will restrict hereafter to these two limiting cases (ER,  $\alpha = 1$ , and BA,  $\alpha = 0$ ). The results obtained for other values of  $\alpha$  will be discussed elsewhere [14]. We advance that they are consistent with the picture described in what follows.

The dynamics is implemented once the network is grown. At the beginning, each individual of the population has the same probability of adopting either of the two available strategies: cooperation ( $s_i = 1$ ) or defection ( $s_i = 0$ ). We let the system evolve for 5000 generations and check whether or not the system has reached a stationary state as given by the fraction,  $c(t)$ , of individuals that are cooperators. We impose that this magnitude is in equilibrium when, taken over a time window of  $10^3$  additional generations, the slope of  $c(t)$  is smaller than  $10^{-2}$  [15]. After such a defined transient time  $t_0$ , we let the system evolve again for  $10^4$  additional time steps, and measure the magnitudes whose behavior is described in the following. All simulations presented hereafter have been carried out for networks made up of 4000 nodes with  $\langle k \rangle = 4$  and results are averaged over at least  $10^3$  different realizations of the networks and initial conditions [16].

The above procedure allows to scrutinize in depth the mi-

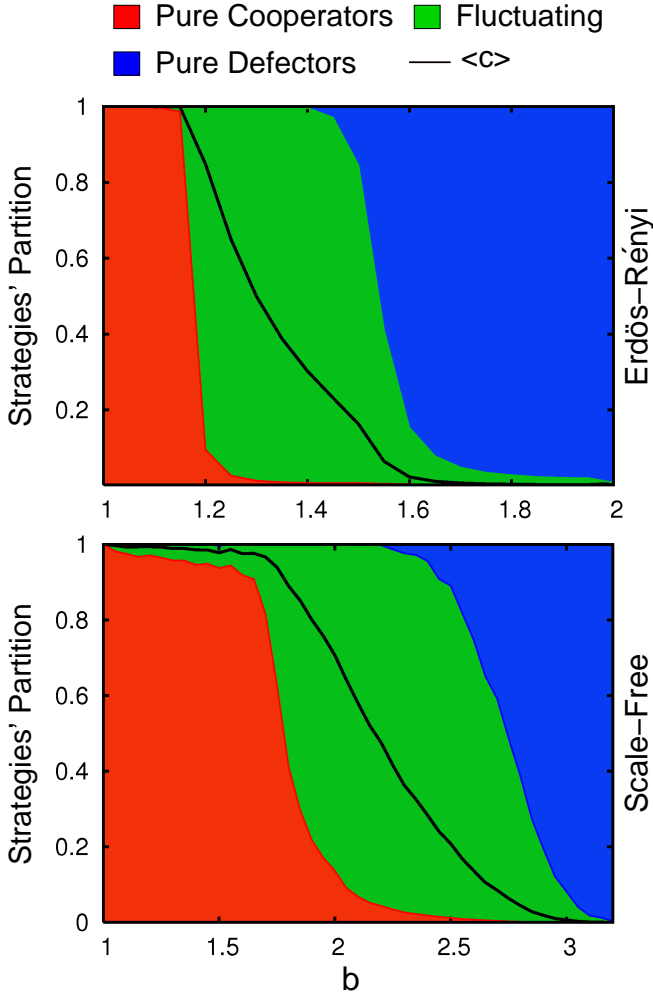


FIG. 1: (color online) Fraction (referred to the total number of individuals of the population) of pure and fluctuating strategies as a function of  $b$ .  $\langle c \rangle$  (black continuous line) represents the asymptotic expected fraction of cooperators at each generation. The border lines separating colored regions correspond to  $\rho_c(b)$  (red-green) and  $1 - \rho_d(b)$  (green-blue). Networks are made up of 4000 nodes and  $\langle k \rangle = 4$ . The exponent of the SF network is  $-3$ .

crossoscopic temporal evolution of cooperation as well as to characterize how its local patterns are formed. Individuals' strategies asymptotically (i.e.  $t > t_0$ ) follow three different behaviors. Let  $P(x, t)$  be the probability that an individual adopts the strategy  $x$  at any time  $t > t_0$ . We say that an element  $i$  of the population is *pure cooperator* if  $P(s_i = 1, t) = 1$ , i.e., it plays as cooperator in all generations after the transient time. Conversely, *pure defectors*, are those individuals for which  $P(s_i = 0, t) = 1$ . A third class is constituted by *fluctuating* individuals, that is, those elements that alternatively spend some time as cooperators and some time as defectors.

Figure 1 shows the densities of the three classes of players as  $b$  is increased, for the two limiting cases of ER (upper) and SF networks (bottom). Note that the fraction of pure cooperators ( $\rho_c$ , continuous leftmost line) is always equal or

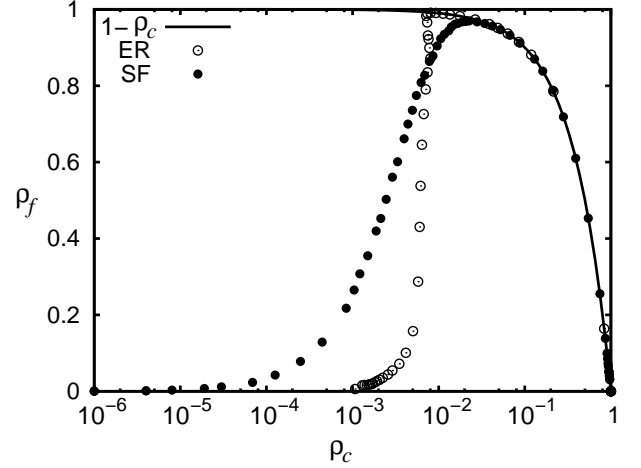


FIG. 2: Fraction of fluctuating strategies as a function of the density of pure cooperators. The solid line is  $\rho_f = 1 - \rho_c$ . Deviations from it means that pure defectors have come into play. Note that the same value of  $\rho_c$  in both topologies corresponds, in general, to different values of  $b$ . Networks parameters are those used in Fig. 1.

smaller than the density  $\langle c \rangle(b)$ , which is the asymptotic expected value of the fraction of cooperators. This indicates that the density of cooperators is, on average, stationary, but not frozen as a significant fraction of individuals are still able to intermittently adopt different strategies. It is observed that in a small range of  $b > 1$ ,  $\rho_c = \langle c \rangle(b)$  for the ER network, while the equality does not hold for any value of  $b$  when the underlying architecture is a SF network. Looking only at pure cooperation, there is a crossover for moderate values of  $b$ . From that point on, the level of pure cooperators in SF networks is above that in ER graphs. Additionally, the decay of  $\rho_c(b)$  is abrupt for homogeneous networks and more smooth for SF ones. Therefore, pure cooperators are more robust to variations of  $b$  in these latter topologies.

Furthermore, there is a region of  $b$  in which almost *all* strategies are fluctuating for the ER graph while this is not the case for heterogeneous networks. This feature is illustrated in Fig. 2, where it is represented the fraction of fluctuating strategies,  $\rho_f$ , as a function of  $\rho_c$ . The deviation from the continuous line, which is the function  $\rho_f = 1 - \rho_c$ , marks the appearance of pure defectors. For both networks, the density of fluctuating elements raises when  $\rho_c$  decreases, however, the decay of  $\rho_f$  is clearly differentiated. While for the SF network this magnitude falls smoothly and well below 1, for the ER network the fraction  $\rho_f$  continues to increase almost to 1, and then decreases suddenly, roughly keeping  $\rho_c$  constant. Moreover, the number of pure cooperators relative to the total number of elements of the population is significantly smaller in the ER networks than in the SF case.

Figure 2 gives even more information about what is going on at a microscopic scale. Why does the fraction  $\rho_c$  is smaller in ER than in SF networks? An important clue comes

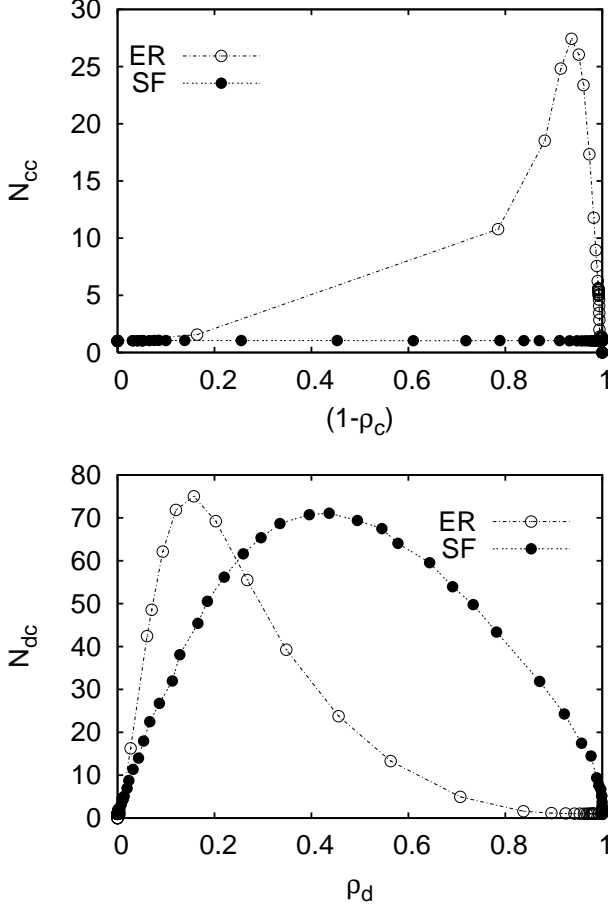


FIG. 3: Number of clusters of pure cooperators (upper panel) and pure defectors (bottom panel) as a function of  $1 - \rho_c$  and  $\rho_d$  in order to have both  $x$  axes growing in the same way as  $b$  does. The figures clearly show that it is possible to have more than one cooperator cluster only for the ER network, while pure defectors are always spread into several cores in SF networks and form a single cluster only in homogeneous structures. Network parameters are those of Fig. 1.

from the analysis of the local distribution of pure cooperators. Let us first define the concept of cluster or core. A cooperator core ( $CC$ ) is a connected component (subgraph) *fully* and *permanently* occupied by cooperator strategy  $s_i = 1$ , i.e., by pure cooperators so that  $P(s_i(t) \neq 1, \forall t > t_0) = 0$ ,  $\forall i \in CC$ . Analogously, a defector core ( $DC$ ) is the subgraph whose elements are pure defectors, namely, when the condition  $P(s_i(t) \neq 0, \forall t > t_0) = 0$ ,  $\forall i \in DC$  is fulfilled. It is easy to see that a  $CC$  cannot be in direct contact with a  $DC$ , but with a cloud of fluctuating elements that constitutes the frontier between these two cores. Note that a  $CC$  is *stable* if none of its elements has a defector neighbor coupled to more than  $k^c/b$  cooperators where  $k^c$  is the number of cooperators linked to the element. Thus, the stability of a  $CC$  is clearly enhanced by a high number of connections among pure cooperators, which implies abundance of cycles in the  $CC$ .

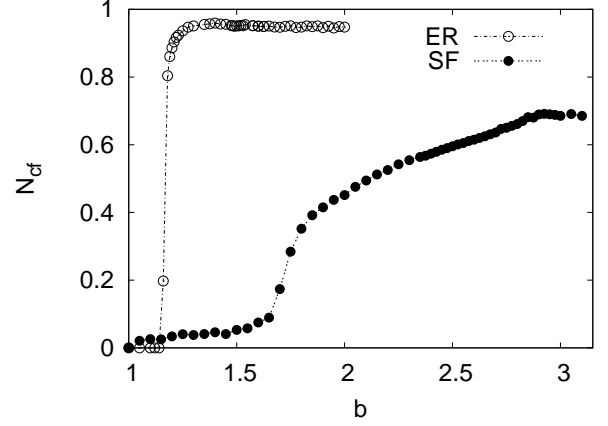


FIG. 4: Fraction of pure cooperators that share at least a link with fluctuating strategists as a function of  $b$ . The high values of  $N_{cf}$  for ER networks is influenced by the existence of several  $CC$ . This is not the case for the SF network where there is only one  $CC$  and  $N_{cf}$  is much smaller. See the text for further details.

This microscopic structure of clusters is at the root of the differences found in the levels of cooperation for both networks and nicely explains why cooperative behavior is more successful in SF networks than in homogeneous graphs. In fact, as far as loops are concerned, the main difference between the two topologies is that the number of small cycles of length  $L$ ,  $N_L$ , are given by [17, 18]  $(\log(N))^L$  and  $(\langle k \rangle - 1)^L$ , respectively. Therefore, it is more likely that SF networks develop a  $CC$  than ER ones. This has been tested numerically by looking at the probability that at least one cooperator core exists. The results [14] indicate that this probability remains 1 for SF networks even for  $b > 2$  and that it approaches zero for large  $b$  following a sort of second order phase transition. On the contrary, for ER networks, the same probability departs from 1 at an intermediate value of  $b$  and shows a sudden jump to 0 at  $b = 2$ , reminiscent of a first order like phase transition.

We next focus on the detailed characterization of  $CC$  and  $DC$  structures. Fig. 3 shows the number of clusters made up of pure cooperators ( $N_{cc}$ , upper panel) and pure defectors ( $N_{dc}$ , bottom panel) for both topologies as a function of  $1 - \rho_c$  and  $\rho_d$ , respectively (recall that  $1 - \rho_c$  grows as  $b$  increases). The first noticeable result concerns the number of cooperator cores. While for ER networks  $N_{cc}$  is equal to 1 only for a small range of  $\rho_c$  values, and later increases up to a maximum, for the SF network the number of such cores is always 1, no matter the value of  $\rho_c$ . That is to say, in one topology (ER), there is a wide region of  $b$  where there are several cooperator cores, whereas pure cooperators in SF networks always form a single core. On the other hand, the behavior of  $N_{cc}$  in SF graphs implies that the cycles discussed above are interconnected, giving rise to more complex structures. We have also verified that the cooperator core in SF networks contains the hubs, which are the ones that stick together the cooperator cycles that would otherwise be disconnected [19].

Looking again at Fig. 3, one realizes that there are also radical differences in the local organization of pure defectors. Again, the structural patterns in both networks can be clearly distinguished. In ER networks, pure defectors first appear distributed in several clusters that later coalesce to form a single core for values of  $b < 2$ , i.e., before the whole system is invaded by defectors. Conversely, defectors are always organized in several clusters for SF networks (except when they occupy the whole system). This latter behavior results from the role hubs play [4]. As they are the most robust against defector's invasion, highly connected individuals survive as pure cooperators until the fraction  $\rho_c$  vanishes (see also Fig. 2), thus keeping around them a highly robust cooperator core that loses more and more elements of its outer layer until cooperation is finally defeated by defection.

The picture emerging from the analyses performed clearly indicates that two different paths characterize the raising (or breakdown) of cooperation. This is also reflected in the way pure cooperators are invaded by defector strategists. Starting at  $b = 1$  all individuals in both topologies are playing as pure cooperators. However, for  $b > 1$ , the pure cooperative level drops below 1 and the population is constituted by pure cooperators as well as by a cloud of fluctuating individuals. When  $b$  is further increased, these fluctuating elements, that constitute the physical frontier of  $CC$  clusters, start to spend more and more time as defectors exploiting cooperators, which ultimately leads to the invasion of the pure cooperators. In this regard, the fragility of a  $CC$  cluster mainly depends on how the cluster's elements are exposed to their frontier. Figure 4 shows the "effective surface" of  $CC$  as given by the fraction of pure cooperators,  $N_{cf}$ , that share at least a link with fluctuating players as a function of  $b$ . It is clear from the figure that in ER networks the invasion is faster than in SF graphs, with a sudden increase of  $N_{cf}$  at low values of  $b$ , signaling the abrupt decay observed in Fig. 1. This is due to the fragmentation of pure cooperators into several  $CC$  that are in a flood of fluctuating elements, which eventually leaves pure cooperators more exposed to invasion. Conversely, in SF networks, the existence of only one  $CC$  makes it more resilient to defection since in this case the number of pure cooperators exposed to fluctuating individuals is much lower, as indicated by  $N_{cf}$ . Hence, the internal cohesion of the  $CC$  cluster sustains cooperation for larger values of  $b$  until an  $N$ -defector core comes out. In summary, we have shown that there are three different classes of individuals according to their asymptotic strategies and that two different patterns of cooperative behavior, determined by the underlying structure, can be clearly identified. Specifically, our results unveil that in SF networks pure cooperators always form a unique cooperator core while several defector clusters coexist. On the contrary, for ER networks, both pure and defector strategists are grouped into several clusters. The microscopic organization of asymptotic cooperation on the evolutionary dynamics of different games, like e.g. Snow-drift, Stag-Hunt [5], can obviously differ from the PD game. However, one should expect that the partition of the population into fluctuating and pure strategists will also generically

hold. Finally, we note that the same structural differences in the emergence and evolution of cooperation has been pointed out in synchronization phenomena on top of complex topologies [20]. Whether or not these common evolutionary patterns that emerge in two distinct phenomena are relevant to explain the ubiquitous presence of SF networks in Nature is still to be tested on more firm grounds. Studies of cooperation on real social networks, like e.g. [21] may help to scale up to a mesoscopic description (in terms of communities) the observations and results presented here.

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